MATHEMATICS

SYMMETRIC KERNEL FUNCTORS AND QUASI-PRIMES

BY

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INTRODUCTION

Let R be a left-Noetherian ring with unity and let P be a prime ideal in R. LAMBEK and MICHLER have shown in Journal of Algebra 25 (1973) that the torsion theory based on the kernel functor $\tau_{R/P}$, induced by the injective hull $E_R(R/P)$, coincides with that defined by the filter of left ideals A such that $[A:r] \cap \mathscr{G}(P)=\emptyset$ for all $r \in R$ where $\mathscr{G}(P)=\{c \in R | rc \in P \to r \in P\}$. In a paper to appear in the Journal of Algebra the authors made use of symmetric kernel functors for which the associated filter has a basis of two sided ideals. These lead to a "symmetric" torsion theory at P defined by the kernel functor σ_{R-P} whose filter has a basis consisting of ideals of R that intersect the m-system $R-P$. The symmetric torsion theories have the advantage that, in the case of a prime ring, they lead to the construction of a structure sheaf on Spec R which preserves many aspects of the commutative theory. The stalks of the structure sheaf are the localizations relative to the kernel functors σ_{R-P}.

The purpose of this paper is to relate the symmetric torsion theory at P to the Lambek-Michler torsion theory and to the prime kernel functors of Goldman. It is shown that σ_{R-P} is equal to $\tau^0_{R/P}$, the largest symmetric kernel functor less than or equal to $\tau_{R/P}$. Quasi-prime kernel functors are defined in such a way that each σ_{R-P} is quasi-prime. This ameliorates the partial correspondence between prime kernel functors and certain prime ideals set up by Goldman.

The main problem faced is to obtain restrictive conditions on quasi-prime kernel functors or on the prime ideal P which will characterize all σ_{R-P} having Goldman's property (T) and hence, in the case of a prime ring, characterizing all T-stalks of the structure sheaf. This problem is only partially solved here but its connection with the question whether the prime kernel functor $\tau_{R/A}$ associated with some critical left ideal $A \in \mathscr{C}'(\sigma_{R-P})$ is symmetric, is made explicit. Here $\mathscr{C}'(\sigma_{R-P})$ is the set of left ideals A maximal with respect to $A \notin \mathscr{T}(\sigma_{R-P})$. The price paid for the many advantages of using the symmetric torsion theory at P is the difficulty of investigating property (T) due to the fact that principal left ideals generated by some $s \in R-P$ are not necessarily in the filter $\mathscr{T}(\sigma_{R-P})$. In order to get information about a quasi-prime σ from the associated primes $\tau_{R/A}$, $A \in \mathscr{C}'(\sigma)$, it would be interesting to know when $\tau^0_{R/A} = \tau^0_{R/P}$ where $P = [A:R]$ is the largest ideal contained in A. Necessary conditions are found on the ring R or on the kernel functor σ in order that $\tau^0_{R/A} = \tau^0_{R/P}$ for every A in $\mathscr{C}'(\sigma)$.

1. Restricted Kernel Functors

Let R be a left-Noetherian ring with unity and denote by $\mathscr{M}(R)$ the category of left R-modules. Let σ be a symmetric kernel functor on $\mathscr{M}(R)$ and let $\mathscr{T}(\sigma)$ be its filter in R. If A is a left ideal and S any subset of R then $[A:S]$ is the left ideal $\{x \in R, xS \subset A\}$. Clearly if B is a left ideal then $[A:B]$ is an ideal and $[A:R]$ is the biggest ideal contained in A. Since σ is symmetric and $[A:S]$ contains every ideal contained in A, it follows that $[A:S]$ is in $\mathscr{T}(\sigma)$ whenever A is. The set of left ideals maximal in the set of left ideals not in $\mathscr{T}(\sigma)$ will be denoted by $\mathscr{C}'(\sigma)$. If $A \in \mathscr{C}'(\sigma)$ then, obviously, R/A is a supporting module for σ and $\tau_{R/A}$ is a prime kernel functor. Since $A \in \mathscr{C}'(\tau_{R/A})$ it follows that A is a critical left ideal, hence $\mathscr{C}'(\sigma)$ consists of critical left ideals. We denote by $\mathscr{C}(\sigma)$ the set of ideals of R maximal in the set of ideals not in $\mathscr{T}(\sigma)$. Every $P \in \mathscr{C}(\sigma)$ is contained in some $A \in \mathscr{C}'(\sigma)$ and if $A \in \mathscr{C}'(\sigma)$ then $[A:R]$ is contained in some element of $\mathscr{C}(\sigma)$. Since the filter $\mathscr{T}(\sigma)$ of a symmetric kernel functor σ is multiplicatively closed it follows that $\mathscr{C}(\sigma)$ consists of prime ideals.

Proposition 1: Let $A \in \mathscr{C}'(\sigma)$ and $s \notin A$, then $B = [A:s] \in \mathscr{C}'(\sigma)$ and $Q_\sigma(R/B) \cong Q_\sigma(R/A)$.

Proof: V. Dlab, [1], proved that a left ideal related to a critical left ideal is critical and that the isomorphism holds. It is easily seen that the left ideal $[A:s]$ is also in $\mathscr{C}'(\sigma)$.

Proposition 2: Let σ be an idempotent kernel functor, then

$$\sigma = \inf \{\tau_{R/A}, A \in \mathscr{C}'(\sigma)\}.$$

Proof: Since $A \in \mathscr{C}'(\sigma)$ implies $\sigma(R/A) = 0$, we have that $\sigma \leqslant \tau_{R/A}$ (the partial ordering on the set of kernel functors is defined by transfer of the inclusion ordering for the corresponding filters). Suppose that τ is an arbitrary kernel functor $\tau \geqslant \sigma$ then $\mathscr{T}(\tau) \supset \mathscr{T}(\sigma)$ and if the inclusion is proper there exists an element $C \in \mathscr{T}(\tau) - \mathscr{T}(\sigma)$. Since R is left-Noetherian, C is contained in some $A \in \mathscr{C}'(\sigma)$ but $C \in \mathscr{T}(\tau)$ implies $A \in \mathscr{T}(\tau)$ or $\tau(R/A) = R/A$. Thus, for this particular A we have that τ is not smaller than $\tau_{R/A}$.

Corollaries:

1. If $P \in \mathscr{C}(\sigma)$ and $P \subset A$, where $A \in \mathscr{C}'(\sigma)$, then $P = \cap [A:s]$, intersection ranging over all $s \notin A$.
2. If $P \in \mathscr{C}(\sigma)$ then $P = \cap A$, intersection ranging over the $A \in \mathscr{C}'(\sigma)$ with $A \supset P$.

3. The direct sum $M = \sum R/A$ of non-isomorphic R/A, $A \in \mathscr{C}'(\sigma)$, induces an idempotent kernel functor τ_M on $\mathscr{M}(R)$ (cf. [2]). Proposition 2 yields $\tau_M = \sigma$.
4. If $\sigma = \sigma_{R-P}$ and every $A' \in \mathscr{C}'(\sigma)$ is of the form $[A:s]$ for some $s \notin A$, $A \in \mathscr{C}'(\sigma)$, then σ is a prime kernel functor.

PROOF: Proposition 1 together with Corollary 3 yield that for all $A \in \mathscr{C}'(\sigma)$ the modules $Q_\sigma(R/A)$ are isomorphic to one another and that σ is a prime kernel functor.

DEFINITIONS: A left ideal $A \in \mathscr{C}'(\sigma)$ is said to be *I-full* for some ideal I of R if $[A+I:R] = I + [A:R]$.

A symmetric kernel functor σ is called a *restricted* kernel functor if $A \in \mathscr{C}'(\sigma)$ yields $[A:R] \in \mathscr{C}(\sigma)$.

A symmetric kernel functor σ is said to be *fibred* if for every $A \in \mathscr{C}'(\sigma)$ there exists a $P \in \mathscr{C}(\sigma)$ and an ideal I in $\mathscr{T}(\sigma)$ such that $[A:I] = P$.

If $\sigma = \sigma_{R-P}$, the symmetric kernel functor associated with the prime ideal P, then $\mathscr{C}(\sigma) = \{P\}$ and hence σ_{R-P} is restricted if and only if $P = \cap A$, $A \in \mathscr{C}'(\sigma)$.

PROPOSITION 3: A symmetric kernel functor is restricted if and only if for each $A \in \mathscr{C}'(\sigma)$ there is a $P \in \mathscr{C}(\sigma)$ such that $[A:R] \subset P$ and A is P-full.

PROOF: Trivially, any $A \in \mathscr{C}'(\sigma)$ is $[A:R]$-full and hence if σ is restricted, A is P-full for some $P \in \mathscr{C}(\sigma)$. Conversely let $A \in \mathscr{C}'(\sigma)$ be P-full where $[A:R] \subset P$. Then $[A+P:R] = P + [A:R] = P$ and hence $A+P \notin \mathscr{T}(\sigma)$ since otherwise $A+P$ would contain an ideal in $\mathscr{T}(\sigma)$, contrary to $[A+P:R] = P$. Since $A \in \mathscr{C}'(\sigma)$ it follows that $P \subset A$ and $P = [A:R]$.

A somewhat stronger form of the fibredness condition yields the following.

PROPOSITION 4: If for every $P \in \mathscr{C}(\sigma)$ and $A \in \mathscr{C}'(\sigma)$ there exists an ideal $I \in \mathscr{T}(\sigma)$ such that $[A:I] = P$ then $\mathscr{C}(\sigma) = \{P\}$, $\sigma = \sigma_{R-P}$ and σ is fibred.

PROOF: Let P, $P_1 \in \mathscr{C}(\sigma)$ and choose $A \in \mathscr{C}'(\sigma)$ such that $P \subset A$, hence $P = [A:R]$. By hypothesis there is an ideal $I \in \mathscr{T}(\sigma)$ such that $P_1 = [A:I] \supset [A:R] = P$, thus $P = P_1$ since both are in $\mathscr{C}(\sigma)$. Thus $\mathscr{C}(\sigma) = \{P\}$, $\sigma = \sigma_{R-P}$, and σ is fibred.

In [5] the ring R was called σ-perfect with respect to a symmetric kernel function σ if for every ideal J of R the extension $J^e = Q_\sigma(R)J$ of J to $Q_\sigma(R)$ is an ideal of $Q_\sigma(R)$. When σ has property (T) it is still necessary to assume that R is σ-perfect in order to obtain the classical properties of the localization $Q_\sigma(R)$. For this reason the following proposition is of interest.

Proposition 5: If R satisfies the Artin-Rees condition: for any two ideals A, B there exists an integer $n>0$ such that $B \cap A^n \subset BA$, then R is σ-perfect for every symmetric kernel functor σ having property (T).

The proof requires two lemmas. (See also [3]).

Lemma 6. If the Artin-Rees condition hold in R then $Q_\sigma(R/A)$ is an R/A-module.

Proof: It is only necessary to show that, in the R-module structure, A annihilates $Q_\sigma(R/A)$. In Goldman's notation elements of $Q_\sigma(R/A)$ can be represented as $[C, f]$ when $C \in \mathscr{T}(\sigma)$, $f \in \mathrm{Hom}_R(C, R/A)$ and $[C, f] = [C', f']$ if f and f' coincide on a left ideal B in $\mathscr{T}(\sigma)$ such that $B \subset C \cap C'$. If $x \in R$ then $x[C, f] = [C', g]$, where C' in $\mathscr{T}(\sigma)$ satisfies $C'x \subset C$, and $g(c') = f(c'x)$ for $c' \in C'$. Choose n so that $C^n \cap A \subset AC$. Since $C^n \in \mathscr{T}(\sigma)$, $x[C, f] = x[C^n, f|C^n] = [C', g]$ where $C'x \subset C^n$. Now if $x \in A$, $C'x \subset C^n \cap A \subset AC$ and for $c' \in C'$ $g(c') = f(c'x) = 0$ since $c'x \in AC$.

Lemma 7. The R/A-module structure of $Q_\sigma(R/A)$ extends uniquely to a ring structure and the R-module homomorphism $\pi_\sigma\colon Q_\sigma(R) \to Q_\sigma(R/A)$, induced by the canonical map $\pi\colon R \to R/A$, is a ring homomorphism.

Proof. The extension of the module structure to a ring structure is done exactly as for $Q_\sigma(R)$, namely if $R_0 = R/\sigma(R)$ and $\xi, \eta \in Q_\sigma(R)$ then $\xi\eta = f_\eta(\xi)$ where $f_\eta\colon Q_\sigma(R) \to Q_\sigma(R)$ is the unique extension of $g_\eta\colon R_0 \to Q_\sigma(R)$ where $g_\eta(x) = x\eta$. Now $\pi_\sigma g_\eta(x) = \pi_\sigma x\eta = x\pi_\sigma\eta = \pi(x)\pi_\sigma(\eta)$, since $Q_\sigma(R/A)$ is an R/A-module. Hence $\pi_\sigma g_\eta(x) = g_{\pi_\sigma\eta}\pi(x)$. It follows that $\pi_\sigma f_\eta = f_{\pi_\sigma\eta}\pi_\sigma$, since they are equal on R_0, and hence $\pi_\sigma(\xi\eta) = \pi_\sigma f_\eta(\xi) = f_{\pi_\sigma\eta}\pi_\sigma\xi = \pi_\sigma\xi\pi_\sigma\eta$.

Proof of Proposition 5: Let A be any ideal of R not in $\mathscr{T}(\sigma)$ and let $A^e = Q_\sigma(R)A$ be its extension to $Q_\sigma(R)$. It was shown in [5] that $A^{ec} = A_\sigma$ where A_σ is the ideal of R defined by $A_\sigma/A = \sigma(R/A)$. Hence R/A_σ is σ-torsion free, $R/A_\sigma \cong i(R)/i(A_\sigma)$ where $i(M) = M/\sigma(M)$, and we have the following commutative diagram with rows exact.

$$\begin{array}{ccccccccc} 0 & \to & i(A_\sigma) & \to & i(R) & \xrightarrow{\beta} & R/A_\sigma & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q_\sigma(A_\sigma) & \to & Q_\sigma(R) & \xrightarrow{\alpha} & Q_\sigma(R/A_\sigma) & \to & 0 \end{array}$$

By Lemma 6 α is a ring homomorphism and $Q_\sigma(A_\alpha) = \ker\alpha$ is an ideal of $Q_\sigma(R)$. Moreover $i(A_\sigma) = \ker\beta = \ker\alpha \cap i(R) = Q_\sigma(A_\sigma) \cap i(R)$ and hence

$$Q_\sigma(R)i(A_\sigma) = Q_\sigma(R)[Q_\sigma(A_\sigma) \cap i(R)] = [Q_\sigma(A_\sigma)]^{ce} = Q_\sigma(A_\sigma).$$

Thus $A^e = A_\sigma^e = Q_\sigma(A_\sigma)$ is an ideal of $Q_\sigma(R)$.

In general the critical left ideals in $\mathscr{C}'(\sigma)$ are not necessarily prime but if σ is restricted then each A in $\mathscr{C}'(\sigma)$ is a prime left ideal. Indeed, if B and C are left ideals of R such that $B \not\subset A$, $C \not\subset A$ but $BC \subset A$, then

$(BR+[A:R])(C+A) \subset A$. Since $BR+[A:R]$ and $C+A$ are in $\mathscr{T}(\sigma)$, this contradicts $A \in \mathscr{C}'(\sigma)$. Thus each $A \in \mathscr{C}'(\sigma)$ for which $[A:R] \subset \mathscr{C}(\sigma)$ is prime.

We can also prove the following.

PROPOSITION 8: Let σ be a symmetric kernel functor having property (T), and suppose that R is σ-perfect, then each $A \in \mathscr{C}'(\sigma)$ is a prime left ideal.

PROOF. Let B, C be left ideals such that $B \not\subset A$, $C \not\subset A$ but $BC \subset A$. Then $(B)C$ A, where (B) is the ideal of R generated by B. Now, $Q_\sigma(R)(B)$ is an ideal of $Q_\sigma(R)$, hence $Q_\sigma(R)(B)C = Q_\sigma(R)B\ Q_\sigma(R)C$ and $(B)^eC^e \subset A^e$. Since A^e is a maximal left ideal of $Q_\sigma(R)$ it follows that if $C^e \not\subset A^e$ then $C^e + A^e = Q_\sigma(R)$ entailing $(B)^e = (B)^eC^e + (B)^eA^e \subset A^e$ and, by contraction, either $(B)_\sigma$ or C_σ is in $A_\sigma = A$, thus either B or C is in A.

COROLLARY 1: Let R be σ-perfect for some symmetric T-functor σ and let $A \in \mathscr{C}'(\sigma)$. If $B \not\subset A$ then $[A:B] = [A:R]$, or, σ is fibred if and only if σ is restricted.

COROLLARY 2: Let σ be a symmetric kernel functor, $A, A' \in \mathscr{C}'(\sigma)$, then $A'B \subset A$ for some left ideal B implies $B \subset A$ or $A' \in \mathscr{C}(\sigma)$. If $B \not\subset A$ then $[A:B] \notin \mathscr{T}(\sigma)$. If for every $A \in \mathscr{C}'(\sigma)$ there is a $P \in \mathscr{C}(\sigma)$ such that $AP \subset A$ then σ is restricted.

The proof is straightforward.

PROPOSITION 9: If σ_{R-P} has property (T), and R is σ_{R-P}-perfect, then the following statements are equivalent:

a. σ_{R-P} is a restricted kernel functor

b. P^e is the Jacobson radical of $Q_\sigma(R)$, where $\sigma = \sigma_{R-P}$.

PROOF: The Jacobson radical $J(Q_\sigma(R))$ is the intersection of the maximal left ideals of $Q_\sigma(R)$, hence property (T) entails $J(Q_\sigma(R)) = \cap A^e$, $A \in \mathscr{C}'(\sigma)$. If $P^e = \cap A^e$ then $P = P^{ec} = (\cap A^e)^c = \cap A$ and thus σ_{R-P} is restricted. Conversely if σ_{R-P} is restricted, $P = \cap A$, $A \in \mathscr{C}'(\sigma)$, then $P^e = (\cap A)^e \subset \cap A^e$ and by contraction, $P = P^{ec} \subset (\cap A^e)^c = \cap A^{ec} = \cap$ $\cap A = P$, thus $P = (\cap A^e)^c$, hence $P^e = \cap A^e = J(Q_\sigma(R))$.

2. QUASI-PRIME KERNEL FUNCTORS

A left R-module M is an R-bimodule if M is a right R-module and $x(my) = (xm)y$ for all m in M and x, y in R.

DEFINITION: An R-bimodule S is called a quasi-support for a kernel functor σ if

1. S is σ-torsion-free as a left R-module.
2. For every nonzero sub-bimodule $S' \subset S$, S/S' is a σ-torsion left R-module.

PROPOSITION 10: If S is a quasi-support for σ, then:

1. S is an essential extension of every nonzero sub-bimodule of S.
2. Every nonzero sub-bimodule of S is a quasi-support for σ.
3. If T is a bimodule such that $T \supset S$, $\sigma(T)=0$ and $\sigma(T/S)=T/S$ then T is also a quasi-support for σ.
4. S is a quasi-support for τ_S, the kernel function associated with S.
5. If S contains a sub-bimodule S' which is a support for σ then S is a support for σ.

PROOF: The proofs of (1)–(4) follow the same lines as the proofs of the corresponding properties of supporting modules (cf. [2]) whereas (5) follows also from [2] since $\sigma(S/S')=S/S'$ and $\sigma(S)=0$.

PROPOSITION 11: Let P be an ideal in R. Then R/P is a quasi-support for σ if and only if $P \in \mathscr{C}(\sigma)$.

PROOF: Suppose R/P is a quasi-support. Then for every ideal I of R such that I properly contains P we have that R/I is σ-torsion and hence $I \in \mathscr{T}(\sigma)$. Since $\sigma(R/P)=0$ it follows that $P \in \mathscr{C}(\sigma)$. Conversely if $P \in \mathscr{C}(\sigma)$, $\sigma(R/P)=0$ and $\sigma(R/I)=R/I$ for every ideal I properly containing P. Hence any sub-bimodule M of R/P has the property that $(R/P)/M$ is σ-torsion and thus R/P is a quasi-support for σ.

DEFINITION: A symmetric kernel functor σ is said to be *quasi-prime* if there is a quasi-support S for σ such that $\sigma=\tau_S^0$ where τ_S^0 is the maximal symmetric kernel functor smaller than τ_S.

EXAMPLE. Let P be a prime ideal in R. The kernel functor σ_{R-P} defined by $\sigma_{R-P}(M)=\{m \in M|(s)m=0, s \notin P\}$ is quasi-prime. Indeed by Proposition 11, R/P is a quasi-support for σ_{R-P}. Suppose $\sigma' > \sigma_{R-P}$ is another symmetric kernel functor. Then there is an ideal $A \in \mathscr{T}(\sigma')$, $A \notin \mathscr{T}(\sigma_{R-P})$ and hence $A \subset P$. This entails $A(R/P)=0$ so that R/P is σ'-torsion and hence $\sigma' \nleq \tau_{R-P}$. Hence $\sigma_{R-P}=\tau_{R/P}^0$. Note that this last statement gives exactly the relation between the symmetric kernel functor σ_{R-P} and the Lambek-Michler torsion theory at a prime ideal P, cf. [4].

PROPOSITION 12: If $E \cong Q_\sigma(R/P)$ for all $P \in \mathscr{C}(\sigma)$ then σ is a quasi-prime kernel functor.

PROOF: If $P \in \mathscr{C}(\sigma)$ then R/P is a quasi-support for σ. Suppose that we have a symmetric kernel functor σ' properly larger then σ, then there is an ideal $A \in \mathscr{T}(\sigma')-\mathscr{T}(\sigma)$. Hence $A \subset P$ for some $P \in \mathscr{C}(\sigma)$ and R/P is then σ'-torsion. We have an exact sequence:

$$0 \to R/P \to Q_\sigma(R/P) \to Q_\sigma(R/P)/(R/P) \to 0$$

where R/P is σ'-torsion and $Q_\sigma(R/P)/(R/P)$ is σ-torsion hence certainly σ'-torsion because $\sigma' > \sigma$. It follows that $Q_\sigma(R/P)$ is σ'-torsion. By the hypothesis, $Q_\sigma(R/P)$ is σ'-torsion for every $P \in \mathscr{C}(\sigma)$, hence σ is the largest symmetric kernel functor for which E is torsion-free or $\sigma = \tau^0_E$. Since $Q_\sigma(R/P)/(R/P)$ is σ-torsion it follows that $\sigma = \tau^0_{R/P}$ and σ is quasi-prime.

PROPOSITION 13: Let σ be an arbitrary symmetric kernel functor. Then $\sigma = \inf \tau^0_{R/P} = (\inf \tau_{R/P})^0$, the inf being taken over ideals $P \in \mathscr{C}(\sigma)$.

PROOF: As before, if σ' is symmetric and $\sigma' > \sigma$ then for some $P \in \mathscr{C}(\sigma)$ we have $\sigma'(R/P) = R/P$ and σ' cannot be smaller than or equal to $\tau^0_{R/P}$. Obviously $\sigma \leqslant \tau^0_{R/P}$ since $\sigma(R/P) = 0$, and σ is symmetric.

COROLLARY 1: If σ, σ' are symmetric kernel functors such that $\sigma' \geqslant \sigma$ and $\sigma'(R/P) \neq R/P$ for all $P \in \mathscr{C}(\sigma)$ then $\sigma = \sigma'$.

COROLLARY 2: Let S be a quasi-support for σ, then for every $\sigma' > \sigma$ such that $\sigma'(S) \neq 0$ we have that $\sigma'(S) = S$.

Indeed, $\sigma'(S)$ is a sub-bimodule in S, hence $S/\sigma'(S)$ is σ-torsion and a fortiori σ'-torsion, thus $S/\sigma'(S) = 0$.

COROLLARY 3: If for all $P \in \mathscr{C}(\sigma)$, the induced symmetric kernel functors $\tau^0_{R/P}$ coincide then $\sigma = \tau^0_{R/P}$, each R/P is a quasi-support for σ, and σ is quasi-prime.

Note, that if S is a quasi-support for σ and if S is also an R-ring, then S contains an isomorphic image of R/P for some $P \in \mathscr{C}(\sigma)$. Indeed, if $f: R \to S$ defines the R-ring structure in S, then $S \supset f(R) \cong R/I$ for some ideal I in R. Since $I \neq R$ we have a nonzero sub-bimodule R/I in S which is then also a quasi-support for σ, hence $I \in \mathscr{C}(\sigma)$ by Proposition 11. This shows that in considering quasi-supports which are R-rings it is sufficient to consider R/P, $P \in \mathscr{C}(\sigma)$, which are *prime* rings.

PROPOSITION 14: Let σ be a symmetric kernel functor, $A \in \mathscr{C}'(\sigma)$ and suppose that $[A:R] = P$ is a prime ideal, then $\tau^0_{R/P} = \tau^0_{R/A}$.

PROOF: Let $0 \neq \bar{x} \in R/P$ and suppose that there is an ideal $I \in \mathscr{T}(\tau^0_{R/A})$ such that $I\bar{x} = 0$, i.e. $I(x) \subset P$ for some $x \notin P$. Hence $I \subset P$ and $I \subset P \subset A$ yields $A \in \mathscr{T}(\tau^0_{R/A}) \subset \mathscr{T}(\tau_{R/A})$, contradicting $\tau_{R/A}(R/A) = 0$. Thus R/P is $\tau^0_{R/A}$-torsion free, or $\tau^0_{R/A} \leqslant \tau_{R/P}$.

Now, let $0 \neq \bar{y} \in R/A$ and suppose there exists an ideal $J \in \mathscr{T}(\tau^0_{R/P})$ such that $J\bar{y} = 0$, i.e. $JRy \subset A$ or $J(A + Ry) \subset A$. But J is $\tau^0_{R/P}$-open while $A + Ry$ is in $\mathscr{T}(\sigma)$ hence in $\mathscr{T}(\tau^0_{R/P})$ since $\sigma(R/P) = 0$, (for if $I \in \mathscr{T}(\sigma)$, $x \notin P$ and $Ix \subset P$ then $I \subset P \subset A$ contradicts $A \notin \mathscr{T}(\sigma)$). Now, since $\tau^0_{R/P}$ is symmetric it follows that $J(A + Ry)$, and hence A, is in $\mathscr{T}(\tau^0_{R/P})$, meaning that $A \supset B$ where B is an ideal in $\mathscr{T}(\tau^0_{R/P})$, then $B \subset [A:R] = P$ yields $P \in \mathscr{T}(\tau^0_{R/P})$ contradicting $\tau^0_{R/P}(R/P) = 0$. Thus we have $\tau^0_{R/P} \leqslant \tau_{R/A}$. Both inequalities yield $\tau^0_{R/P} = \tau^0_{R/A}$.

Corollary 1: If σ has property (T) and R is σ-perfect, then all $A \in \mathscr{C}'(\sigma)$ are prime left ideals and hence $\tau^0_{R/A} = \tau^0_{R/P}$ since $P = [A:R]$ is obviously also a prime ideal.

Corollary 2: If σ is restricted then for each $A \in \mathscr{C}'(\sigma)$, $[A:R] \in \mathscr{C}(\sigma)$ and again $\tau^0_{R/A} = \tau^0_{R/P}$, $P = [A:R]$.

Corollary 3: Let σ be a restricted kernel functor such that $\tau_{R/A}$ is symmetric for every $A \in \mathscr{C}'(\sigma)$. If the $\tau^0_{R/P}$ coincide for all $P \in \mathscr{C}(\sigma)$ then σ is prime.

Proposition 15: Let $\sigma = \sigma_{R-P}$, P a prime ideal in R. If for all $A \in \mathscr{C}'(\sigma)$ we have $P \notin \mathscr{T}(\tau^0_{R/A})$ then $\sigma = \tau^0_{R/A}$ for all $A \in \mathscr{C}'(\sigma)$.

Proof: If $0 \neq \bar{x} \in \tau^0_{R/A}(R/P)$ then $I(x) \subset P$ for some ideal $I \in \mathscr{T}(\tau^0_{R/A})$. Hence $I \subset P$ and $P \in \mathscr{T}(\tau^0_{R/A})$, a contradiction. Thus $\tau^0_{R/A} \leqslant \tau_{R/P}$. The proof of $\tau^0_{R/P} \leqslant \tau_{R/A}$ is exactly the same as the proof of this inequality in Proposition 14, using only that $[A:R] \subset P$ instead of $[A:R] = P$, (or use $\sigma = \inf \tau_{R/A}$, $A \in \mathscr{C}'(\sigma)$).

Since σ is quasi-prime with quasi-support R/P inducing σ, we have $\sigma = \tau^0_{R/P} = \tau^0_{R/A}$ for all $A \in \mathscr{C}'(\sigma)$.

Corollary 1: If $\sigma = \sigma_{R-P}$ then $P \notin \mathscr{T}(\tau^0_{R/A})$, $A \in \mathscr{C}'(\sigma)$ if and only if $\sigma = \tau^0_{R/A}$.

Corollary 2: If σ_{R-P} is a prime kernel functor then $P \notin \mathscr{T}(\tau_{R/A})$ for every $A \in \mathscr{C}'(\sigma)$. The converse is true when $\tau_{R/A}$ is symmetric for all $A \in \mathscr{C}'(\sigma)$.

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